Journal of Computational and Applied Mathematics 230 (2009) 607–613

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# An implicit Landweber method for nonlinear ill-posed operator equations☆

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ARTICLE INFO

*Article history:*
Received 10 October 2007
Received in revised form 1 October 2008

*Keywords:*
Ill-posed problem
Landweber iteration
Implicit method

ABSTRACT

In this paper, we are interested in the solution of nonlinear inverse problems of the form $F(x) = y$. We propose an implicit Landweber method, which is similar to the third-order midpoint Newton method in form, and consider the convergence behavior of the implicit Landweber method. Using the discrepancy principle as a stopping criterion, we obtain a regularization method for ill-posed problems. We conclude with numerical examples confirming the theoretical results, including comparisons with the classical Landweber iteration and presented modified Landweber methods.



## 1. Introduction

Many mathematical physics problems can be reduced to solve the nonlinear problem

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \to Y$, with $D(F) \subset X$. Due to the nonlinearity of $F$, the solution of problem (1.1) may not be unique. Usually, the map $F$ is compact and (1.1) is an ill-posed problem. We call the equation ill-posed if the solution of Eq. (1.1) does not depend continuously on the data $y$. Throughout this paper, we assume that $y^\delta$ are the available approximate data with

$$\|y^\delta - y\| \le \delta, \tag{1.2}$$

where $\delta$ denotes the noise level.

Then, a numerically stable and reliable approximation can merely be obtained by the usage of regularization techniques (see [1,2]). In contrast to Tikhonov regularization, several iteration methods for nonlinear operators were under investigation during the last years. In the paper of Hanke et al. [3] the well known Landweber iteration for linear ill-posed problems has been extended to the nonlinear case

$$x_{k+1}^\delta = x_k^\delta - F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta), \quad k = 0, 1, \ldots. \tag{1.3}$$

The Landweber method is easy to be realized numerically. But it usually needs a large number of iteration steps, in particular if the error $\delta$ is small. In [4], J. Xu et al. introduced the Šamanskii's technique to the Landweber iteration and defined a frozen Landweber iteration

$$\begin{aligned} &x_{k+1,i}^\delta = x_{k,i}^\delta - F'(x_k^\delta)^*(F(x_{k,i}^\delta) - y^\delta), \quad i = 0, 1, \ldots, m-1, \\ &x_k^\delta = x_{k,0}^\delta, x_{k+1}^\delta = x_{k,m}^\delta, \quad k = 0, 1, \ldots. \end{aligned} \tag{1.4}$$

☆ This work is supported by the NSF of PR China (No. 40774056).

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0377-0427/$ – see front matter 
doi:10.1016/j.cam.2008.12.016

The numerical results show that frozen Landweber method needs shorter running time compared with classical Landweber iteration.

In order to improve the behavior of classical Landweber iteration, Li, et al. used Runge–Kutta methods to numerically solve the continuous method (see [5]) and obtained R–K type Landweber method [6]

$$x_{k+1}^{\delta} = x_k^{\delta} - F'\left(x_k^{\delta} - \frac{1}{2}F'(x_k^{\delta})^*(F(x_k^{\delta}) - y^{\delta})\right)^* \left[\left(F(x_k^{\delta} - \frac{1}{2}F'(x_k^{\delta}))^*(F(x_k^{\delta}) - y^{\delta}) - y^{\delta}\right)\right] \quad k = 0, 1, \ldots. \tag{1.5}$$

In this paper, we consider an implicit Landweber method

$$\begin{aligned} &x_{k+1}^{\delta} = x_k^{\delta} + hK(x_k^{\delta}, h), \\ &K(x, h) = -F'(x + \frac{1}{2}hK(x, h))^*(F(x) - y^{\delta}), \quad k = 0, 1, \ldots. \\ &\text{i.e} \\ &x_{k+1}^{\delta} = x_k^{\delta} - hF'\left(\frac{1}{2}(x_k^{\delta} + x_{k+1}^{\delta})\right)^* (F(x_k^{\delta}) - y^{\delta}), \quad k = 0, 1, \ldots \end{aligned} \tag{1.6}$$

which is obtained from the third-order midpoint Newton method [7] by replacing $F'(\cdot)^{-1}$ with $F'(\cdot)^*$. We wish that method (1.6) would have some advantages over explicit method due to its numerical stability.

In Section 2, we prove that the implicit Landweber method (1.6), combined with a suitable stopping criterion, is a regularization method. The numerical implementation of the implicit method is studied in Section 3. In Section 4 the proposed iteration is applied to solve the nonlinear ill-posed convolution problems and parameter identification problems. The numerical results show effectiveness of this method.

## 2. Convergence of implicit Landweber method

In this section, a convergence analysis for the implicit Landweber iteration (1.6) is considered. The convergence analysis follows essentially the scheme of [6,8,9] and requires that $F$ satisfies the following assumptions, where $\Omega \subset D(F)$ is an open neighborhood of $x_*$:

A1. there exists $M > 0$ such that $\|F'(x)\| \le M$ for $x \in \Omega$;

A2. $F'$ is uniformly Lipschitzian in $\Omega$ with Lipschitz constant $L$;

A3. there exists $0 < \eta < \frac{1}{2}$ such that

$$\| F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x})\| \le \eta\|F(x) - F(\tilde{x})\|, \quad \forall x, \tilde{x} \in \Omega.$$

The assumption A3 guarantees that for all $x, \tilde{x} \in \Omega$,

$$\frac{1}{1+\eta}\|F'(x)(x - \tilde{x})\| \le \|F(x) - F(\tilde{x})\| \le \frac{1}{1-\eta}\|F'(x)(x - \tilde{x})\| \tag{2.1}$$

holds.

Under these assumptions, and assuming $x_0$ close enough to $x_*$, we will prove that the sequence $\{x_k\}$ converges to a solution $x_*$ of (1.1) when $\delta = 0$. On the other hand, for $\delta \ne 0$, we will show that the condition

$$\| F(x_{k_*}^{\delta}) - y^{\delta}\| \le \tau\delta \le \|F(x_k^{\delta}) - y^{\delta}\|, \quad \forall k \le k_* - 1 \tag{2.2}$$

can be satisfied within a finite number of iterations, with $\tau$ being a positive number depending $\eta$ of A3,

$$\tau > \frac{1+\eta}{1-2\eta} > 2. \tag{2.3}$$

Before we state convergence and stability results for the implicit Landweber iteration, we consider the implicit method

$$K(x, h) = -F'\left(x + \frac{1}{2}hK(x, h)\right)^* (F(x) - y). \tag{2.4}$$

**Lemma 2.1** (*[10]*). *Let $F'$ be continuous and satisfy a Lipschitz condition with constant $L$. Then there exists a unique solution of (2.4), which can be obtained by simple iteration.*

*Moreover, let $K(x, 0) = -F'(x)^*(F(x) - y)$, we have*

$$\|K(x, h)\| \le \frac{\|K(x, 0)\|}{1 - hL/2\|F(x) - y\|}, \quad x \in \Omega. \tag{2.5}$$

**Proof.** In fact, after eventually diminishing $h$, we can get

$$\|K(x, h) - K(x, 0)\| \le \frac{Lh}{2}\|K(x, h)\|\|F(x) - y\|. \quad \square$$

**Lemma 2.2.** *Let $x_*$ be a solution of (1.1). Suppose that assumptions A1 and A2 are satisfied. Assume that (2.5) holds and*
(a)$R_0 := \|x_0 - x_*\|$;
(b)$B(x_*, \hat{R}_0) \subset \Omega$, *with* $\hat{R}_0 = R_0 + hM(\|F(x_0) - y\| + 2MR_0)/[2 - hL(\|F(x_0) - y\| + 2MR_0)]$.
*We put* $z_k = \phi(x_k)$, *where*
$\phi(x) := x + \frac{1}{2}hK(x, h)$.
*If* $x_k \in B(x_*, R_0)$, *there exists* $h_1 > 0$ *such that* $z_k \in \Omega$, $i = 1, \ldots, s$, *for* $\forall h \leq h_1$.

**Proof.** If $x_k \in B(x_*, R_0)$, by (2.5) we can get

$$\|z_k - x_k\| \leq \frac{h}{2}\|K(x, h)\| \leq \frac{hM\|F(x_k) - y\|}{2 - hL\|F(x_k) - y\|}.$$

From assumption A1 we obtain

$$\|z_k - x_*\| \leq \|x_k - x_*\| + \frac{hM(\|F(x_0) - y\| + 2MR_0)}{2 - hL(\|F(x_0) - y\| + 2MR_0)},$$

i.e. $z_k \in \Omega$. □

**Remark 2.3.** In view of $\phi(x_k) \in \Omega$, the properties A1–A3 for $z_k = \phi(x_k)$ also hold. Therefore if $x_k^\delta \in B(x_*, R_0)$, the properties A1–A3 for $z_k^\delta$ also hold.

**Theorem 2.4.** *Suppose the assumptions of Lemma 2.2 hold, for* $0 \leq k < k_*$. *Denote by* $k_*(\delta)$ *the termination index of the iteration according to the stopping rule (2.2), with* $\tau$ *satisfying (2.3). For any* $0 \leq k < k_*$, *then there exists* $h_1 > 0$ *such that for* $\forall h \leq h_1$, *we have*

$$\|x_{k+1}^\delta - x_*\| \leq \|x_k^\delta - x_*\|.$$

*and, if* $\delta = 0$ *we have*

$$\sum_{k=0}^{\infty} \|F(x_k) - y\|^2 \leq \infty. \tag{2.6}$$

**Proof.** Denote $e_k = x_k^\delta - x_*$. From (1.6) it follows that

$$\begin{aligned} \|e_{k+1}\|^2 &\leq \|e_k\|^2 - 2h(e_k, F'(z_k^\delta)^*(F(x_k^\delta) - y^\delta)) + h^2M^2\|F(x_k^\delta) - y^\delta\|^2 \\ &\leq \|e_k\|^2 + 2h(F(x_k^\delta) - y^\delta - F'(z_k^\delta)e_k, F(x_k^\delta) - y^\delta) + (h^2M^2 - 2h)\|F(x_k^\delta) - y^\delta\|^2. \end{aligned} \tag{2.7}$$

By (2.5) we can get

$$\|K(x_k^\delta, h)\| \leq \frac{M}{\gamma}\|F(x_k^\delta) - y^\delta\|,$$

with $\gamma = 1 - \frac{hL}{2}(\|F(x_0) - y\| + 2MR_0)$.

Moreover, assumptions A2 and A3 yield

$$\begin{aligned} \|F(x_k^\delta) - y^\delta - F'(z_k^\delta)e_k\| &\leq \|y - y^\delta\| + \|F(x_k^\delta) - y - F'(x_k^\delta)e_k\| + \|(F'(z_k^\delta) - F'(x_k^\delta))e_k\| \\ &\leq \delta + \eta\|F(x_k^\delta) - y\| + L\|z_k^\delta - x_k^\delta\|\|e_k\| \\ &\leq (1 + \eta)\delta + \eta\|F(x_k^\delta) - y^\delta\| + \frac{hLR_0}{2}\|K\| \\ &\leq (1 + \eta)\delta + \left(\eta + \frac{hLMR_0}{\gamma}\right)\|F(x_k^\delta) - y^\delta\|, \end{aligned} \tag{2.8}$$

$$\|e_{k+1}\|^2 \leq \|e_k\|^2 + h\left(hM^2 - 2 + 2\eta + \frac{2(1 + \eta)}{\tau} + \frac{2hLMR_0}{\gamma}\right)\|F(x_k^\delta) - y^\delta\|^2. \tag{2.9}$$

There exists $h_1 > 0$ such that $\gamma > 0$ and

$$hM^2 + \frac{2hLMR_0}{\gamma} \leq 1, \quad \forall h \leq h_1. \tag{2.10}$$

For the stopping criterion (2.2) it follows from (2.9) that

$$\|x_{k+1}^\delta - x_*\|^2 \leq \|x_k^\delta - x_*\|^2 + \left(2\eta - 1 + \frac{2(\eta + 1)}{\tau}\right)h\|F(x_k^\delta) - y^\delta\|^2.$$

By (2.3) we have

$$\|x_{k+1}^{\delta} - x_*\| \le \|x_k^{\delta} - x_*\|.$$

In the case of noise free data ($\delta = 0$) from (2.7) and (2.8) we can get

$$\|x_{k+1}^{\delta} - x_*\|^2 + (1 - 2\eta)h\|F(x_k) - y\|^2 \le \|x_k - x_*\|^2, k \in N, \quad \forall h \le h_1.$$

By induction we obtain

$$h\sum_{k=0}^{\infty}\|F(x_k) - y\|^2 \le \frac{1}{1-2\eta}\|x_0 - x_*\|^2, \tag{2.11}$$

and assertion (2.6) follows. □

We remark that if $\delta \neq 0$, then we can show in a similar way that

$$h\sum_{k=0}^{k_*-1}\|F(x_k^{\delta}) - y^{\delta}\|^2 \le \frac{\tau}{(1-2\eta)\tau - 2(1+\eta)}\|x_0^{\delta} - x_*\|^2. \tag{2.12}$$

**Theorem 2.5.** *Let $\delta = 0$ in (1.2). If $F$ satisfies A1 and A2, $x_*$ is solvable, then $x_k$ converges to $x_* \in B(x_0, R_0)$. Moreover, if $x^{\dagger}$ denotes the unique solution of minimal distance to $x_0$, and if, in addition, $N(F'(x^{\dagger})) \subseteq N(F'(x))$ for all $x \in B(x_0, R_0)$, then $x_k$ converges to $x^{\dagger}$.*

**Proof.** Let $\widetilde{x}_*$ be any solution of (1.1) in $B(x_0, R_0)$, and set $e_k := x_k - \widetilde{x}_*$.

Next, we show that $e_k$ is a Cauchy sequence. For $j \ge k$ we choose $l$ with $j \ge l \ge k$ such that

$$\|y - F(x_l)\| \le \|y - F(x_i)\|, k \le i \le j.$$

We have

$$\|e_j - e_k\| \le \|e_j - e_l\| + \|e_l - e_k\|,$$

and

$$\begin{aligned}
&\|e_j - e_l\|^2 = 2(e_l - e_j, e_l) + \|e_j\|^2 - \|e_l\|^2,\\
&\|e_l - e_k\|^2 = 2(e_l - e_k, e_l) + \|e_k\|^2 - \|e_l\|^2.
\end{aligned} \tag{2.13}$$

From Theorem 2.4 it follows that, for $k \to \infty$, the last two terms on each of the right-hand side of (2.13) converge to 0.

From A1, A2 and (2.1) it follows

$$\begin{aligned}
|(e_l - e_k, e_l)| &\le h\left|\sum_{r=k}^{l-1}(F'(z_r)(x_l - \widetilde{x}_*), F(x_r) - y)\right|\\
&\le h\sum_{r=k}^{l-1}(\|F'(x_r)(x_l - \widetilde{x}_*)\| + \|(F'(z_r) - F'(x_r))(x_l - \widetilde{x}_*)\|)\|F(x_r) - y\|\\
&\le h\sum_{r=k}^{l-1}(\|F'(x_r)(x_l - x_r + x_r - \widetilde{x}_*)\| + LR_0\|z_r - x_r\|)\|F(x_r) - y\|\\
&\le h\sum_{r=k}^{l-1}\|F(x_r) - y\|\left[\left(2(1+\eta) + \frac{hLMR_0}{\gamma}\right)\|F(x_r) - y\| + (1+\eta)\|F(x_l) - y\|\right]\\
&\le h\left[3(1+\eta) + \frac{hLMR_0}{\gamma}\right]\sum_{r=k}^{l-1}\|F(x_r) - y\|^2.
\end{aligned}$$

Thus, by Theorem 2.4, for $k \to \infty$, the second term of the right-hand side in (2.13) tends to zero. Similarly, we can get

$$|(e_l - e_j, e_l)| \to 0, j \to \infty.$$

With these estimates, we have shown that $e_k$ and thus $x_k$ are Cauchy sequences. We denote by $x_*$ the limit of $x_k$ and observe that $x_*$ is a solution of (1.1).

We can prove the uniqueness of the solution according to the method of [3]. □

**Theorem 2.6.** *Under the assumptions of Theorem 2.5, when $\delta$ fulfills (1.2) if the implicit Landweber method (1.6) is stopped at $k_*(\delta)$, according to the stopping criterion (2.2) and (2.3), then*

$$x_{k_*(\delta)}^{\delta} \to x_*, \delta \to 0.$$

**Proof.** Consider a sequence $\delta_n$ such that $\delta_n \to 0$ (as $n \to \infty$), and let $y_n := y^{\delta_n}$ be a corresponding sequence of perturbed data. Denote $k_n := k_*(\delta_n)$ the corresponding stopping index determined by the discrepancy principle (2.2), (2.3). Let us examine two possible behaviors of the sequence $k_n$.

Firstly, we suppose that $k$ is a finite accumulation point of $k_n$. Without loss of generality we can assume that $k_n = k$ for all $n \in N$. Then we have

$$\|y_n - F(x_k^{\delta_n})\| \le \tau \delta_n.$$

Since $k$ is fixed, $x_k^\delta$ depends continuously on $y^\delta$ and $\lim_{n\to\infty} x_k^{\delta_n} = x_k$, where $x_k$ denotes the $k$-th iterate computed by method (1.6) with $\delta = 0$; therefore, we also have

$$0 = F(x_k^{\delta_n}) - y^{\delta_n} \to F(x_k) - y, n \to \infty,$$

i.e. $x_k = x_*$ and $\lim_{k\to\infty} x_k^{\delta_n} \to x_*$.

As the second case, we consider $k_n \to \infty$, $n \to \infty$. Without loss of generality, we assume that $k_n$ increases monotonically with $n$. Then, for $n > m$ we conclude from Theorem 2.4,

$$\|x_{k_n}^{\delta_n} - x_*\| \le \|x_{k_{n-1}}^{\delta_n} - x_*\| \le \cdots \le \|x_{k_m}^{\delta_n} - x_*\| \le \|x_{k_m}^{\delta_n} - x_{k_m}\| + \|x_{k_m} - x_*\|. \tag{2.14}$$

From Theorem 2.5 we note that we can fix $m$ so large that the last term on the right-hand side of (2.14) is sufficiently close to zero; now that $k_m$ is fixed, we can conclude that the left-hand side of (2.14) must go to zero as $n \to \infty$, and the proof is complete. □

## 3. Numerical implementation of the implicit method

In this section, we discuss the numerical implementation of the implicit method, including choices of step size.

For the implicit Landweber method, by using simple iteration we can obtain the following iteration

$$\begin{aligned} &x_{k+1}^\delta = x_k^\delta + h_k \widetilde{K}, \\ &K^{j+1} = -F'(x_k^\delta + \frac{1}{2} h_k K^j)^*(F(x_k^\delta) - y^\delta), \quad j = 1, 2 \dots m; \\ &\widetilde{K} = K^m, \quad k = 0, 1, \dots, \end{aligned} \tag{3.1}$$

with $K^1 = -F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta)$.

For $h_k \equiv 1$ and $m = 1$ it is just the classical Landweber iteration.

For the sake of simplification, we choose $m = 2$ to test the convergence of a concrete modified Landweber iteration

$$x_{k+1}^\delta = x_k^\delta - h_k F'\left(x_k^\delta - \frac{1}{2} h_k F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta)\right)^* (F(x_k^\delta) - y^\delta). \tag{3.2}$$

In fact, for $h_k \equiv 1$, let $\phi(x) = x - \frac{1}{2} F'(x)^*(F(x) - y)$, if $x \in B(x_0, R_0)$, then $\phi(x) \in B(x_0, R_0)$.

Similarly, we can obtain $\phi(x_k^\delta) \in B(x_0, R_0)$, and then the properties A1–A3 for $\phi(x_k^\delta)$ also hold (see [6]).

Thus, analogously to the proof of theorem one can verify, if

$$M^2 - 2 + 2\eta + \frac{2(1+\eta)}{\tau} + \frac{LMR_0}{2} \le 0,$$

then

$$\|x_{k+1}^\delta - x_*\| \le \|x_k^\delta - x_*\|.$$

In the same way, we can verify that the iteration (3.2) is a regularization method.

Furthermore, we consider the adaptive variable step size in (3.2). We put

$$h_k := \min\left\{\frac{1}{2\|F'(x_k^\delta)\|^2}, \frac{1}{(\|F'(x_k^\delta)\| + LR_0 M/4)^2}\right\},$$

where $R_0 := \|x_0 - x_*\|$, and $M, L$ are positive numbers in the assumptions A1 and A2.

We note the first part guaranteed that $\phi(x_k^\delta) \in B(x_0, R_0)$ and the latter guaranteed the convergence behavior of the modified Landweber method (3.2). In the following section, we can see the choice of adaptive variable step size is effective.

## 4. Numerical results

In order to verify the effectiveness of the method, we solve several problems frequently used as test problems in literature. Here, we discuss the results obtained for the nonlinear ill-posed convolution problem and some parameter identification problems. Moreover, we numerically compare the implicit Landweber iteration with the classical Landweber iteration (1.3) and presented modified Landweber methods (Frozen Landweber (1.4) ($m = 2$) and R–K type Landweber methods (1.5)).

**Table 1**
The results ($e_{k_*}(k_*)$) of Example 4.1 with $x_0 = 1.3$.

| $\delta$ | Implicit Landweber | Classical Landweber | R–K Landweber | Frozen Landweber |
|---|---|---|---|---|
| 1.e−2 | 0.1417(128) | 0.1727(247) | 0.1407(104) | 0.1471(73) |
| 1.e−3 | 0.0728(3064) | 0.0958(4431) | 0.0721(3025) | 0.0763(1641) |
| 1.e−4 | 0.0611(16,385) | 0.0856(18,715) | 0.0604(16,308) | 0.0649(8400) |
| 1.e−5 | 0.0609(32,708) | 0.0855(35,138) | 0.0602(32,627) | 0.0647(16,572) |
| $T_{k=1000}$ | 3.7969 | 1.9063 | 3.9219 | 2.0625 |

**Table 2**
The results ($e_{k_*}(k_*)$) of Example 4.1 with $x_0 = 1.5$.

| $\delta$ | Implicit Landweber | Classical Landweber | R–K Landweber | Frozen Landweber |
|---|---|---|---|---|
| 1.e−2 | 0.1627(198) | 3.51(1,615) | 0.1623(194) | 0.1912(162) |
| 1.e−3 | 0.0878(3954) | 3.51(16,656) | 0.0873(3925) | 0.1112(2637) |
| 1.e−4 | 0.0777(17,968) | 3.51(33,942) | 0.0765(17,921) | 0.1020(9964) |
| 1.e−5 | 0.0769(34,362) | 3.51(51,265) | 0.0764(34,312) | 0.1019(18,195) |

In all of the numerical tests considered below, we report some results with different random noises. The parameters are chosen by experience via various numerical examples. Here $k_*$ is the stopping steps of iterations required to satisfy the discrepancy principle with $\tau = 2.01$, and $x^\delta_{k_*}$ denotes the approximate solution. $T$ is the CPU time, whose unit is second.

**Example 4.1.** The problem under consideration is the solution of nonlinear ill-posed convolution problem, with the operator $F$ given by

$$\begin{aligned} &F : L^2[0,1] \to L^2[0,1] \\ &x \mapsto \int_0^s x(s-t)x(t)\mathrm{d}t. \end{aligned} \tag{4.1}$$

In [2] it was proven that this operator is Frechét-differentiable with a Lipschitz-continuous derivative. Moreover, it was verified that $F$ satisfied A1 and A3 locally. Thus the general results of Section 2 are applicable.

For numerical computations, we assume that the exact solution is $x_*(t) = 1$, and choose initial element $x_0(t) = 1.3$ (see Table 1) or $x_0(t) = 1.5$ (see Table 2).

In Tables 1 and 2, we compare the implicit Landweber iteration $h = 1$ with the classical Landweber iteration and presented modified Landweber iterations (R–K Landweber and Frozen Landweber methods). We show the relative error $e_{k_*}$, the stopping step $k_*$ according to the stopping rule (2.2) and $T_{k=1000}$ which is the CPU running time when stopping step $k = 1000$.

We can see that the three modified methods indeed improve the behavior of classical Landweber iteration. In particular, the implicit Landweber and R–K Landweber show their superiority in the relative error $e_{k_*}$, the Frozen Landweber method in the CPU running time. So, we consider the implicit Landweber with variable step size and hope that the proposed method has advantages over the two modified methods mentioned above.

**Example 4.2.** This example is taken from [8]. Given the boundary value problem

$$\begin{aligned} &-(au_x)_x = f, \\ &u(0) = g_0, u(1) = g_1. \end{aligned} \tag{4.2}$$

This parameter estimation problem can be formulated in terms of a nonlinear operator equation

$$F(a) = u^0,$$

where $F$ is the operator which maps a conductivity $a \in X := \{a \in H^1(0,1) : a(t) \geq \epsilon > 0\}$ (here, $\gamma$ denotes a fixed positive constant) onto its indirect measurement $u^0 \in L^2(0,1)$.

In [1] it was proven that this operator is Fréchet-differentiable with a Lipschitz-continuous derivative; moreover it was verified that $F$ satisfies conditions A1 and A2.

The function $f$ is given by $f(t) = -e^t(\hat{a}(t)+\pi\cos(2\pi t))+(e-1)\pi\cos(2\pi t)$, with $\hat{a}(t) = 1+\frac{1}{2}\sin(2\pi t)$. The solution of (4.2) for $a = \hat{a}(t)$ is $\hat{u}(t) = e^t+(1-e)t-1$. We used initial guesses of the form $a_0 = \hat{a}(t)+220t^2(1-t)^2(0.25-t)(0.75-t)$. The differential equations were solved with linear splines on a uniform grid with 10 grid points.

Table 3 shows the results of the classical Landweber iteration and three modified Landweber methods, with $h = 1$ in implicit Landweber method. Table 4 shows the results of the implicit Landweber method (3.1) with the adaptive variable step size. Compared with each other, the CPU time of implicit Landweber method is much shorter in the same situation.

**Example 4.3.** Here, we consider the same parameter identification problem as in previous example with $\hat{a}(t) = 1$. The numerical solution with Tikhonov regularization has been considered in [11]. Here, the exact data is $\hat{u}(t) = t(t-1)$ and

**Table 3**
The results ($e_{k_*}(k_*)$) of Example 4.2.

| $\delta$ | Implicit Landweber | Classical Landweber | R–K Landweber | Frozen Landweber |
|---|---|---|---|---|
| 1.e−2 | 0.7180(453) | 0.7153(451) | 0.7174(453) | 0.7133(224) |
| 1.e−3 | 0.2491(8756) | 0.2495(8688) | 0.2483(8754) | 0.2500(4317) |
| 1.e−4 | 0.1104(66,299) | 0.1104(66,165) | 0.1104(66,021) | 0.1104(33,055) |
| $T_{k=1000}$ | 11.25 | 6.047 | 11.91 | 8.234 |

**Table 4**
The results of the implicit Landweber with variable step size for Example 4.2.

| $\delta$ | $k_*$ | $e_{k_*}$ | $T$ |
|---|---|---|---|
| 1.e−2 | 60 | 0.7176 | 2.359 |
| 1.e−3 | 1222 | 0.2494 | 13.78 |
| 1.e−4 | 9242 | 0.1104 | 141.8 |

**Table 5**
The results of the implicit Landweber with variable step size for Example 4.3.

| $\delta$ | $k_*$ | $e_{k_*}$ | $e_{k_*}/(\delta^{1/2})$ |
|---|---|---|---|
| 1.e−2 | 7 | 9.22e−2 | 0.922 |
| 1.e−3 | 36 | 6.04e−2 | 1.910 |
| 1.e−4 | 569 | 3.61e−2 | 3.610 |
| 1.e−5 | 9 331 | 5.60e−3 | 1.771 |
| 1.e−6 | 24 917 | 6.78e−4 | 0.678 |

accordingly the right-hand side of the differential equation is 2. For

$$a_0(t) = 1 + 0.1\left(9 - 4t + 4t^2 - 4\frac{\cosh(t) + \cosh(t-1)}{\sinh 1}\right),$$

$a_0 - a_* \in R(F'(a^*))^*$.

Table 5 shows the results of the implicit Landweber (3.1) with the adaptive variable stepsize. From the last column, it can be seen that the iterates are convergent with a convergence rate $O(\delta^{1/2})$. Concerning the convergence rate of the implicit Landweber iteration, we will touch upon it in our next research.

## 5. Conclusions

In this paper, we have proposed the implicit Landweber method for nonlinear ill-posed operator equations. The method's convergence properties are obtained, and the performance of the method is testified by numerical examples. The numerical results show that the implicit Landweber method is feasible, widely convergent and faster.

However the implicit method requires proper numerical implementation (e.g. simple iteration, Newton iteration etc.). So it is an expected work to construct different modified Landweber methods.

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